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Pointwise simultaneous approximation by combinations of Bernstein operators☆

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Abstract

We prove in the present paper some new direct and inverse results on pointwise simultaneous approximation by combinations of Bernstein operators by using $\omega^{2r}_{\varphi^\lambda}(f,t)$, where $\omega^{2r}_{\varphi^\lambda}(f,t)$ is the Ditzian–Totik modulus of smoothness. We also give an equivalent relation on pointwise approximation by these operators using $\omega^{2r}_{\varphi^\lambda}(f,t)$ when $0 \leqslant \lambda < 1 - 1/r$.

Keywords: Pointwise simultaneous approximation; Combination of Bernstein operators; Moduli of smoothness

1. Introduction

For Bernstein operators on $C[0,1]$ defined by

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) \equiv \binom{n}{k} x^k (1-x)^{n-k}.$$

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Ditzian [4] gave the following interesting direct estimate:

$$| B_n(f,x) - f(x) | \leqslant C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right), \quad x \in [0,1], \tag{1}$$

where $0 \leqslant \lambda \leqslant 1$, $\varphi(x) = (x(1-x))^{1/2}$, and the $2r$th Ditzian–Totik modulus of smoothness is

$$\omega^{2r}_{\varphi^\lambda}(f,t) = \sup_{0<h\leqslant t}\ \sup_{0\leqslant x\leqslant 1} | \Delta^{2r}_{h\varphi^\lambda(x)} f(x) |,$$

in which $\Delta^{2r}_t f(x) = \sum_{j=0}^{2r}(-1)^j \binom{2r}{j} f(x+(r-j)t)$ when $rt \leqslant x \leqslant 1-rt$ and $\Delta^{2r}_t f(x) = 0$, otherwise.

The inequality (1) unified the classical estimate ($\lambda = 0$) and the norm estimate developed by Ditzian and Totik ($\lambda = 1$). It follows from [3,11] that the inverse result to (1) is true (see also Theorem A with $r = 1$). Such results for polynomial approximation were previously investigated in [5,6].

Since Bernstein operators cannot be used in the investigation of higher orders of smoothness, Butzer [1] introduced combinations of Bernstein operators. Ditzian and Totik [7, p. 116] (see also [2, p. 278]) extended this method and defined the combinations as

$$B_n(f,r,x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f,x),$$

where n_i and $C_i(n)$ satisfy the following conditions:

$$\text{(a) } n = n_0 < \cdots < n_{r-1} \leqslant An; \quad \text{(b) } \sum_{i=0}^{r-1} | C_i(n) | \leqslant C;$$
$$\text{(c) } \sum_{i=0}^{r-1} C_i(n) = 1; \quad \text{(d) } \sum_{i=0}^{r-1} C_i(n) n_i^{-\rho} = 0,\ \rho = 1,2,\ldots,r-1. \tag{2}$$

Concerning with the pointwise approximation by combinations of Bernstein operators, Zhou [12] obtained direct and inverse results for these operators by using the rth classical modulus of smoothness in 1995. Moreover, Guo et al. [8] in 2000 and the author [10] in 1999 proved another direct and inverse results for these operators by using the $2r$th Ditzian–Totik modulus of smoothness. The main result of Guo et al. [8] or Xie and Xie [10] is

Theorem A. *Let $r \in N$, $1 - \frac{1}{r} \leqslant \lambda \leqslant 1$ and $0 < \alpha < 2r$. Then for all $f \in C[0,1]$ we have*

$$B_n(f,r,x) - f(x) = O\left(\left(n^{-1/2}\varphi^{1-\lambda}(x)\right)^\alpha\right) \text{ as } n \to \infty$$
$$\Leftrightarrow \omega^{2r}_{\varphi^\lambda}(f,t) = O\left(t^\alpha\right) \text{ as } t \to +0. \tag{3}$$

Moreover, if $r \geqslant 2$ and $0 \leqslant \lambda < 1 - \frac{1}{r}$, (3) does not hold.

We note that in the case of $\lambda = 1$ the above result can also be founded in [2].

In this paper, we investigate the relation between the rate of convergence for the derivatives of combinations of Bernstein operators and the smoothness for the derivatives of functions. Thus, we prove new direct and inverse results on pointwise simultaneous approximation by those combinations of Bernstein operators. Our main result can be stated as

Theorem 1. *Let $r, s \in N$, $0 \leqslant \lambda \leqslant 1$ and $s < \alpha < s + \frac{2r}{2-\lambda}$. Then for all $f^{(s)} \in C[0,1]$*

$$\frac{n^s(n-s)!}{n!} B_n^{(s)}(f,r,x) - f^{(s)}(x) = O\left(\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^{\alpha-s}\right) \text{ as } n \to \infty$$
$$\Leftrightarrow \omega_{\varphi^\lambda}^{2r}\left(f^{(s)}, t\right) = O\left(t^{\alpha-s}\right) \text{ as } t \to +0, \tag{4}$$

where $\delta_n(x) = \varphi(x) + n^{-1/2}$.

Remark. The example of the end of Section 2 shows that Theorem 1 does not hold when $\alpha > s + \frac{2r}{2-\lambda}$, while for $\alpha = s + \frac{2r}{2-\lambda}$ (4) remains open.

We note that (3) does not hold when $r \geqslant 2$ and $0 \leqslant \lambda < 1 - \frac{1}{r}$. Naturally, it is interesting to know what happens to the relation between the rate of convergence for these combinations of Bernstein operators and the smoothness of functions if $r \geqslant 2$ and $0 \leqslant \lambda < 1 - \frac{1}{r}$. In this paper, we also consider this problem and obtain an equivalent result on pointwise approximation by these operators using $\omega_{\varphi^\lambda}^{2r}(f,t)$ when $r \geqslant 2$ and $0 \leqslant \lambda < 1 - \frac{1}{r}$.

Throughout this paper, C denotes a positive constant independent of n and x, which may be a different constant in different cases. The organization of the paper is as follows: in Section 2, we first provide a direct result (see Theorem 2). Using this theorem, we are able to prove the implication $\Leftarrow$ in Theorem 1. Section 3 deals with the inverse result (see Theorem 3). Theorem 3 is actually stronger than the implication $\Rightarrow$ in Theorem 1.

2. Direct result

In this section we give a direct result which is of its own significance. Using this result, we can prove the implication $\Leftarrow$ in Theorem 1. We state our direct result as follows:

Theorem 2. *Let $r, s \in N$, $0 \leqslant \lambda \leqslant 1$ and $J = \max\{j : r\lambda - 2r + j \leqslant 0,\ j \leqslant 2r - 1\}$. Then for all $f^{(s)} \in C[0,1]$, and $n \in N$ with $n \geqslant M$, we have*

$$\left|\frac{n^s(n-s)!}{n!} B_n^{(s)}(f,r,x) - f^{(s)}(x)\right|$$
$$\leqslant C\left(\sum_{i=r}^{J} \omega^i\left(f^{(s)}, \left(n^{-r}\varphi^{2(i-r)}(x)\right)^{1/i}\right) + \omega_{\varphi^\lambda}^{2r}\left(f^{(s)}, n^{-1/2}\varphi^{1-\lambda}(x)\right)\right.$$
$$\left. + n^{-r}\varphi^{2r(1-\lambda)}(x) \parallel f^{(s)} \parallel \right), \tag{5}$$

where M is a positive constant and $\omega^i(f,t)$ is the ith classical modulus of smoothness.

To prove Theorem 2 we should use the K-functionals given by

$$K_{2r,\varphi^\lambda}(f,t^{2r}) = \inf\left\{\| f-g \| + t^{2r} \| \varphi^{2r\lambda} g^{(2r)} \|_{\infty}:\ g^{(2r-1)} \in A.C_{.\text{loc}}\right\}$$

and

$$K_i(f,t^i) = \inf\left\{\| f-g \| + t^i \| g^{(i)} \|_{\infty}:\ g^{(i-1)} \in A.C_{.\text{loc}}\right\},$$

where $\|\cdot\|_\infty = \|\cdot\|_{L_\infty[0,1]}$. Following Ditzian and Totik [7, p. 11], there exist constants $C > 0$ and $t_0 > 0$, such that for $0 < t < t_0$,

$$C^{-1} K_{2r,\varphi^\lambda}(f,t^{2r}) \leqslant \omega^{2r}_{\varphi^\lambda}(f,t) \leqslant C K_{2r,\varphi^\lambda}(f,t^{2r}) \tag{6}$$

and

$$C^{-1} K_i(f,t^i) \leqslant \omega^i(f,t) \leqslant C K_i(f,t^i). \tag{7}$$

We need also the following expressions of derivatives of B_n given by simple calculations (see e.g. [7, p. 125]):

$$B_n^{(s)}(f,x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \int_0^{1/n} \cdots \int_0^{1/n} f^{(s)}\left(\frac{k}{n} + \sum_{i=1}^{s} u_i\right) du_1 \cdots du_s\, p_{n-s,k}(x),$$

where $f^{(s)} \in C[0,1]$. For the sake of convenience, we introduce the auxiliary operators for $n \geqslant s$, $s \in N$ and $g \in C[0,1]$

$$B_{n,s}(g,x) = n^s \sum_{k=0}^{n-s} \int_0^{1/n} \cdots \int_0^{1/n} g\left(\frac{k}{n} + \sum_{i=1}^{s} u_i\right) du_1 \cdots du_s\, p_{n-s,k}(x)$$

and the combinations of these operators

$$B_{n,s}(g,r,x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i,s}(g,x),$$

where n_i and $C_i(n)$ satisfy (2). Obviously, $B_{n,s}(g,x)$ and $B_{n,s}(g,r,x)$ are also linear and bounded operators on $C[0,1]$, and $B_{n,s}(1,r,x) = 1$. Moreover, for $f^{(s)} \in C[0,1]$ there holds

$$B_{n,s}\left(f^{(s)},x\right) = \frac{n^s (n-s)!}{n!} B_n^{(s)}(f,x) \quad \text{and}$$

$$B_{n,s}\left(f^{(s)},r,x\right) = \frac{n^s (n-s)!}{n!} B_n^{(s)}(f,r,x).$$

By a simple computation and introduction, we have the following:

Lemma 2.1. *If $s, l \in N$ and $n \geqslant s$, then*

$$B_{n,s}\left((t-x)^l, x\right) = \sum_{m=0}^{\left[\frac{l}{2}\right]} q_{m,l}(x)\left(1-\frac{s}{n}\right)^{\left[\frac{l}{2}\right]-m} \frac{\varphi(x)^{2\left[\frac{l}{2}\right]-2m}}{n^{l-\left[\frac{l}{2}\right]+m}}, \tag{8}$$

where $q_{m,l}(x)$ are fixed polynomials in x, which do not depend on n.

Using (8) and (2)(d), we obtain immediately the following:

Lemma 2.2. *Let $r, s, l \in N$ and $n \geqslant s$. Then there hold*

(a) $B_{n,s}\left((t-x)^l, r, x\right) = 0$ *for* $r \geqslant 2, l = 1, \ldots, r-1$ *and* $x \in [0,1]$, (9)

(b) $\mid B_{n,s}((t-x)^r, r, x) \mid \leqslant M_1 n^{-r}$ *for* $x \in [0,1]$, (10)

(c) $\mid B_{n,s}\left((t-x)^l, r, x\right) \mid \leqslant M_2 n^{-r} \varphi^{2l-2r}(x)$ *for* $l = r+1, \ldots, 2r$ *and* $x \in E_n = \left[\frac{1}{n}, 1-\frac{1}{n}\right]$, (11)

(d) $\mid B_{n,s}((t-x)^{2r}, x) \mid \leqslant M_3 n^{-r} \varphi^{2r}(x)$ *for* $x \in E_n$, *and* (12)

(e) $\mid B_{n,s}((t-x)^{2r}, x) \mid \leqslant M_4 n^{-2r}$ *for* $x \in E_n^c = \left[0, \frac{1}{n}\right) \cup \left(1-\frac{1}{n}, 1\right]$, (13)

where M_i $(i = 1, 2, 3, 4)$ is a positive constant independent of n and x.

Using the same approach as in [8, p. 111] one can easily obtain

Lemma 2.3. *If $r \in N$, $r \geqslant 2$, $\frac{1}{r} < \lambda \leqslant 1$ and $f^{(2r-1)} \in A.C_{.\text{loc}}$, then for $m < r\lambda$*

$$\| \varphi^{2r\lambda-2m} f^{(2r-m)} \|_\infty \leqslant C\left(\| f \| + \| \varphi^{2r\lambda} f^{(2r)} \|_\infty\right). \tag{14}$$

To prove Theorem 2 we need also

Lemma 2.4. *Let $r, s \in N$, $0 \leqslant \lambda \leqslant 1$ and $J = \max\{j : r\lambda - 2r + j \leqslant 0, j \leqslant 2r-1\}$. Then for $f^{(2r-1)} \in A.C_{.\text{loc}}$ and $x \in E_n$, we have*

$$\mid B_{n,s}(f, r, x) - f(x) \mid \leqslant C\left(\sum_{i=r}^{J} \omega^i\left(f, \left(n^{-r}\varphi^{2(i-r)}(x)\right)^{1/i}\right) + n^{-r}\varphi^{2r(1-\lambda)}(x)\left(\| f \| + \| \varphi^{2r\lambda} f^{(2r)} \|_\infty\right)\right), \tag{15}$$

where $n \geqslant \max\left\{\left(2JM_2^{1/r}\right)^{J/r}, s\right\}$.

Proof. We should construct a new operator $A_n(f,x)$ that satisfies $A_n((t-x)^j,x)=0$ for $j=1,\ldots,J$ by adding operators to $B_{n,s}(f,r,x)$. For $n\geqslant s$ and $r\leqslant i\leqslant J$, we define

$$T_{n,i}(f,x)=\begin{cases} -\frac{1}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right)\vec{\Delta}^{i}_{|R_{n,i}(x)|^{1/i}}f(x) & \text{for } 0\leqslant x\leqslant \frac{1}{2},\\ \frac{(-1)^{i+1}}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right)\vec{\Delta}^{i}_{-|R_{n,i}(x)|^{1/i}}f(x) & \text{for } \frac{1}{2}<x\leqslant 1,\end{cases}$$

where $R_{n,i}(x)=B_{n,s}((t-x)^i,r,x)$, and $\vec{\Delta}^{i}_{t}f(x)=\sum_{k=0}^{i}(-1)^k\begin{pmatrix} i\\ k\end{pmatrix}f(x+(i-k)t)$ for $0\leqslant x\leqslant 1-it$ and $\vec{\Delta}^{i}_{t}f(x)=0$ otherwise. Using (3.2.3) and (2.1.4) in [7], one can easily show that

$$|\,T_{n,i}(f,x)\,|\leqslant C\omega^i\left(f,|\,R_{n,i}(x)\,|^{1/i}\right) \tag{16}$$

and for $n\geqslant\left(2JM_2^{1/r}\right)^{J/r}$

$$T_{n,i}\left((t-x)^j,x\right)=\begin{cases}0, & j<i,\\ -R_{n,i}(x), & j=i,\\ C_{i,j}\left(\operatorname{sgn} R_{n,i}(x)\right)|\,R_{n,i}(x)\,|^{j/i}, & j>i,\end{cases} \tag{17}$$

where $C_{i,j}$ is a constant that depends on i and j but not on n and x. Basing on the second line in (17), we can add the operator defined above to $B_{n,s}(f,r,x)$ to eliminate the i moment $R_{n,i}(x)$. However, for $i<j\leqslant J$,

$$T_{n,i}\left((t-x)^j,x\right)=C_{i,j}\left(\operatorname{sgn} R_{n,i}(x)\right)|\,R_{n,i}(x)\,|^{j/i}\ .$$

To eliminate the term $T_{n,i}\left((t-x)^j,x\right)$ for $j=j_1>i$, we add the operator

$$T_{n,i,j_1}(f,x)=\frac{-C_{i,j_1}}{j_1!}\left(\operatorname{sgn} R_{n,i}(x)\right)\vec{\Delta}^{j_1}_{|R_{n,i}(x)|^{1/i}}f(x)$$

(and a similar version for $\frac{1}{2}<x\leqslant 1$). Obviously, for $j_1<j\leqslant J$ we still have $T_{n,i,j_1}((t-x)^j,x)=C_{i,j_1,j}\left(\operatorname{sgn} R_{n,i}(x)\right)|\,R_{n,i}(x)\,|^{j/i}$, which we eliminate by adding $T_{n,i,j_1,j_2}(f,x)$ for $j=j_2>j_1$ given in a similar way. In general we should define $T_{n,i,j_1,\ldots,j_k}(f,x)$ by induction. We have

$$T_{n,i,j_1,\ldots,j_{k-1}}((t-x)^{j_k},x)=C_{i,j_1,\ldots,j_k}\left(\operatorname{sgn} R_{n,i}(x)\right)|\,R_{n,i}(x)\,|^{j_k/i}$$

and for $i<j_1<\cdots<j_k\leqslant J$ we define

$$T_{n,i,j_1,\ldots,j_k}(f,x)=\frac{-C_{i,j_1,\ldots,j_k}}{j_k!}\left(\operatorname{sgn} R_{n,i}(x)\right)\vec{\Delta}^{j_k}_{|R_{n,i}(x)|^{1/i}}f(x)$$

together with an appropriate modification for $\frac{1}{2}<x\leqslant 1$. Hence, we still have

$$|\,T_{n,i,j_1,\ldots,j_k}(f,x)\,|\leqslant C\omega^{j_k}(f,|\,R_{n,i}(x)\,|^{1/i})\leqslant C\omega^i\left(f,|\,R_{n,i}(x)\,|^{1/i}\right) \tag{18}$$

and for $n \geqslant \left(2JM_2^{1/r}\right)^{J/r}$

$$T_{n,i,j_1,\dots,j_k}\left((t-x)^j, x\right) = \begin{cases} 0, & j<j_k, \\ -C_{i,j_1,\dots,j_k}\left(\operatorname{sgn} R_{n,i}(x)\right)|R_{n,i}(x)|^{j_k/i}, & j=j_k, \\ C_{i,j_1,\dots,j_k,j}\left(\operatorname{sgn} R_{n,i}(x)\right)|R_{n,i}(x)|^{j/i}, & j>j_k. \end{cases} \quad (19)$$

Now we define the new operator

$$A_n(f,x) = B_{n,s}(f,r,x) + \sum_{i=r}^{J}\left\{T_{n,i}(f,x) + \sum_{r\leqslant i<j_1<\cdots<j_k\leqslant J} T_{n,i,j_1,\dots,j_k}(f,x)\right\},$$

where the second sum is taken on all finite sequences $j_1,\dots,j_k$, for which $i<j_1<\cdots<j_k\leqslant J$.

Obviously, $A_n(f,x)$ is a linear and bounded operator on $C[0,1]$, and $A_n(1,x)=1$. Using (9), (17) and (19), we have, for $j=1,\dots,J$, $A_n((t-x)^j,x)=0$. Using (10), (11), (17) and (19), we have, for $j=J+1,\dots,2r-1$,

$$\begin{aligned} \left| A_n\left((t-x)^j,x\right)\right| &\leqslant C\left(n^{-r}\varphi^{2j-2r}(x) + \sum_{i=r}^{J}\left| R_{n,i}(x)\right|^{j/i}\right) \\ &\leqslant C\left(n^{-r}\varphi^{2j-2r}(x) + \sum_{i=r}^{J}\left(n^{-r}\varphi^{2i-2r}(x)\right)^{j/i}\right) \\ &\leqslant Cn^{-r}\varphi^{2j-2r}(x). \end{aligned} \quad (20)$$

Using Taylor expansion, we can write that

$$\begin{aligned} A_n(f,x) - f(x) &= \sum_{j=J+1}^{2r-1}\frac{1}{j!}A_n((t-x)^j,x)f^{(j)}(x) \\ &\quad + \frac{1}{(2r-1)!}A_n\left(\int_x^t (t-u)^{2r-1}f^{(2r)}(u)\,du, x\right) \\ &\equiv I_1 + I_2. \end{aligned} \quad (21)$$

For $r=1$ or $0\leqslant\lambda\leqslant\frac{1}{r}$, $J=2r-1$. Then there is not I_1 in (21). For $r\geqslant 2$ and $\frac{1}{r}<\lambda\leqslant 1$, we conclude from (20) and (14)

$$\begin{aligned} |I_1| &\leqslant C\sum_{j=J+1}^{2r-1} n^{-r}\varphi^{2j-2r}(x)\,|f^{(j)}(x)| \\ &\leqslant Cn^{-r}\varphi^{2r(1-\lambda)}(x)\left(\|f\| + \|\varphi^{2r\lambda}f^{(2r)}\|_\infty\right). \end{aligned} \quad (22)$$

To estimate I_2 for $0 \leqslant \lambda \leqslant 1$, we use (2.8) in [8, p. 112] and (12) to obtain

$$\begin{aligned}&\left| B_{n,s}\left(\int_x^t (t-u)^{2r-1} f^{(2r)}(u)\,du, r, x\right)\right| \\ &\quad \leqslant \sum_{i=0}^{r-1} |\ C_i(n)\ ||\ \varphi^{2r\lambda} f^{(2r)}\ \|_\infty\ B_{n_i,s}\left(\frac{(t-x)^{2r}}{\varphi^{2r\lambda}(x)}, x\right) \\ &\quad \leqslant C n^{-r} \varphi^{2r(1-\lambda)}(x)\ \|\ \varphi^{2r\lambda} f^{(2r)}\ \|_\infty .\end{aligned}$$

By (2.8) in [8, p. 112], (10) and (11), we get

$$\begin{aligned}\left| T_{n,i}\left(\int_x^t (t-u)^{2r-1} f^{(2r)}(u)\,du, x\right)\right| &\leqslant C\varphi^{-2r\lambda}(x)\ |\ R_{n,i}(x)\ |^{2r/i} \|\ \varphi^{2r\lambda} f^{(2r)}\ \|_\infty \\ &\leqslant C n^{-r} \varphi^{2r(1-\lambda)}(x)\ \|\ \varphi^{2r\lambda} f^{(2r)}\ \|_\infty .\end{aligned}$$

Similarly, we have

$$\left| T_{n,i,j_1,\dots,j_k}\left(\int_x^t (t-u)^{2r-1} f^{(2r)}(u)\,du, x\right)\right| \leqslant C n^{-r} \varphi^{2r(1-\lambda)}(x)\ \|\ \varphi^{2r\lambda} f^{(2r)}\ \|_\infty .$$

Therefore, we obtain the following:

$$|\ I_2\ | \leqslant C n^{-r} \varphi^{2r(1-\lambda)}(x)\ \|\ \varphi^{2r\lambda} f^{(2r)}\ \|_\infty . \tag{23}$$

Combining (21)–(23), we obtain

$$|\ A_n(f,x) - f(x)\ | \leqslant C n^{-r} \varphi^{2r(1-\lambda)}(x) \left(\|\ f\ \| + \|\ \varphi^{2r\lambda} f^{(2r)}\ \|_\infty\right).$$

Hence, using (16), (18), (10) and (11), we conclude finally

$$\begin{aligned}|B_{n,s}(f,r,x) - f(x)\ | &\leqslant\ |\ A_n(f,x) - f(x)\ | + \sum_{i=r}^{J} \Bigg\{ |\ T_{n,i}(f,x)\ | \\ &\qquad + \sum_{r \leqslant i < j_1 < \cdots < j_k \leqslant J} |\ T_{n,i,j_1,\dots,j_k}(f,x)\ | \Bigg\} \\ &\leqslant C\left(\sum_{i=r}^{J} w^i\left(f, \left(n^{-r}\varphi^{2(i-r)}(x)\right)^{1/i}\right)\right. \\ &\qquad \left. + n^{-r}\varphi^{2r(1-\lambda)}(x)\left(\|\ f\ \| + \|\ \varphi^{2r\lambda} f^{(2r)}\ \|_\infty\right)\right).\end{aligned}$$

The proof of Lemma 2.4 is now complete. □

Proof of Theorem 2. For $x \in E_n$, we choose $g_n \equiv g_{n,x,\lambda}$ (see (6)) such that

$$\begin{aligned}&\|\ f^{(s)} - g_n\ \| + \left(n^{-1/2}\varphi^{1-\lambda}(x)\right)^{2r}\ \|\ \varphi^{2r\lambda} g_n^{(2r)}\ \|_\infty \\ &\quad \leqslant 2C\omega_{\varphi^\lambda}^{2r}\left(f^{(s)}, n^{-1/2}\varphi^{1-\lambda}(x)\right).\end{aligned} \tag{24}$$

From the definition of $B_{n,s}$ and (15), we have for $n \geqslant \max\left\{\left(2JM_2^{1/r}\right)^{J/r}, s\right\}$

$$\left|\frac{n^s(n-s)!}{n!}B_n^{(s)}(f,r,x) - f^{(s)}(x)\right|$$

$$\leqslant C \parallel f^{(s)} - g_n \parallel + \mid B_{n,s}(g_n, r, x) - g_n(x) \mid$$

$$\leqslant C\left(\sum_{i=r}^{J} \omega^i\left(g_n, \left(n^{-r}\varphi^{2(i-r)}(x)\right)^{1/i}\right) + \parallel f^{(s)} - g_n \parallel\right.$$

$$\left. + n^{-r}\varphi^{2r(1-\lambda)}(x)\left(\parallel g_n \parallel + \parallel \varphi^{2r\lambda} g_n^{(2r)} \parallel_\infty\right)\right)$$

$$\leqslant C\left(\sum_{i=r}^{J} \omega^i\left(f^{(s)}, \left(n^{-r}\varphi^{2(i-r)}(x)\right)^{1/i}\right) + \parallel f^{(s)} - g_n \parallel\right.$$

$$\left. + n^{-r}\varphi^{2r(1-\lambda)}(x) \parallel \varphi^{2r\lambda} g_n^{(2r)} \parallel_\infty + n^{-r}\varphi^{2r(1-\lambda)}(x) \parallel f^{(s)} \parallel\right),$$

which, together with (24), yields (5) for $x \in E_n$.

It remains to prove (5) for $x \in E_n^c$. By (7) we have g_n satisfying

$$\parallel f^{(s)} - g_n \parallel + n^{-r} \parallel g_n^{(r)} \parallel_\infty \leqslant 2C\omega^r\left(f^{(s)}, n^{-1}\right). \tag{25}$$

Using Taylor expansion, (9), (13), and Cauchy–Schwartz inequality, we obtain

$$\mid B_{n,s}(g_n, r, x) - g_n(x) \mid = \frac{1}{(r-1)!}\left|B_{n,s}\left(\int_x^t (t-u)^{r-1} g_n^{(r)}(u)\,\mathrm{d}u, r, x\right)\right|$$

$$\leqslant C \parallel g_n^{(r)} \parallel_\infty \sum_{i=0}^{r-1} \mid C_i(n) \mid B_{n_i,s}\left(\mid t-x \mid^r, x\right)$$

$$\leqslant Cn^{-r} \parallel g_n^{(r)} \parallel_\infty .$$

Thus, it follows from (25) that

$$\left|\frac{n^s(n-s)!}{n!}B_n^{(s)}(f,r,x) - f^{(s)}(x)\right| \leqslant C\left(\parallel f^{(s)} - g_n \parallel + n^{-r} \parallel g_n^{(r)} \parallel_\infty\right)$$

$$\leqslant C\omega^r\left(f^{(s)}, n^{-1}\right),$$

which implies (5) for $x \in E_n^c$. The proof of Theorem 2 is complete. □

To prove the implication $\Leftarrow$ in Theorem 1 we need the following.

Lemma 2.5. *If $r \in N$, $0 \leqslant \lambda \leqslant 1$ and $0 < \beta < \frac{2r}{2-\lambda}$, then $\omega^{2r}_{\varphi^\lambda}(f,t) = O\left(t^\beta\right)$ as $t \to +0$ implies*

$$\omega^i(f,t) = O\left(t^{\beta(1-\lambda/2)}\right) \quad \text{as } t \to +0 \quad \text{for } i = r, \ldots, 2r. \tag{26}$$

Proof. First let $i = 2r$. Hence, it follows from (3.1.5) in [7] that

$$\omega^{2r}(f,t) \leqslant C\omega^{2r}_{\varphi^\lambda}\left(f, t^{1-\lambda/2}\right) \leqslant Ct^{\beta(1-\lambda/2)}.$$

Assuming that (26) is valid for $i = j+1, \ldots, 2r$, we may use the Marchaud inequality (see [7, (4.3.1)]) to obtain for $i = j$

$$\begin{aligned}\omega^j(f,t) &\leqslant Ct^j\left(\int_t^c \frac{\omega^{j+1}(f,u)}{u^{j+1}}\,\mathrm{d}u + \| f \|\right)\\ &\leqslant Ct^j\int_t^c u^{\beta(1-\lambda/2)-j-1}\,\mathrm{d}u + Ct^j \| f \| \leqslant Ct^{\beta(1-\lambda/2)}.\end{aligned}$$

From these, (26) follows by induction. The proof of Lemma 2.5 is complete. □

Proof of the implication $\Leftarrow$ in Theorem 1. We use simply (5) and (26) to have

$$\begin{aligned}&\left|\frac{n^s(n-s)!}{n!}B_n^{(s)}(f,r,x) - f^{(s)}(x)\right|\\ &\quad\leqslant C\left(\sum_{i=r}^{J}\left(n^{-r}\varphi^{2(i-r)}(x)\right)^{(\alpha-s)(1-\lambda/2)/i} + \left(n^{-1/2}\varphi^{1-\lambda}(x)\right)^{\alpha-s}\right)\\ &\quad\leqslant C\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^{\alpha-s},\end{aligned}$$

which verifies the implication $\Leftarrow$ in Theorem 1. □

At the end of this section we show that the implication $\Leftarrow$ in (4) does not hold for $\alpha > s + \frac{2r}{2-\lambda}$. In fact, for $f(x) = \frac{x^{r+s}}{(r+s)\cdots(r+1)}$, using Taylor expansion, (9) and (8), we have for $x = \frac{1}{n}$

$$\begin{aligned}\left|\frac{n^s(n-s)!}{n!}B_n^{(s)}(f,r,x) - f^{(s)}(x)\right| &= \mid B_{n,s}\left(t^r, r, x\right) - x^r \mid\\ &= \mid B_{n,s}\left((t-x)^r, x\right) \mid \geqslant Cn^{-r}.\end{aligned}$$

However, $\omega^{2r}_{\varphi^\lambda}\left(f^{(s)}, t\right) = 0$. Let $x = \frac{1}{n}$, then

$$\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^{\alpha-s} \leqslant 2^{(1-\lambda)(\alpha-s)}n^{-(1-\lambda/2)(\alpha-s)} = o\left(n^{-r}\right),$$

which shows that the left-hand equality in (4) does not hold for $\alpha > s + \frac{2r}{2-\lambda}$.

3. Inverse result

In this section we establish an inverse theorem, which is stronger than the implication ⇒ in Theorem 1. This inverse theorem is

Theorem 3. *Let $r, s \in N$, $0 \leqslant \lambda \leqslant 1$ and $s < \alpha < s + 2r$. Then for all $f^{(s)} \in C[0,1]$*

$$\frac{n^s (n-s)!}{n!} B_n^{(s)}(f, r, x) - f^{(s)}(x) = O\left(\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^{\alpha - s}\right) \text{ as } n \to \infty$$

implies $\omega_{\varphi^\lambda}^{2r}\left(f^{(s)}, t\right) = O\left(t^{\alpha - s}\right)$ *as* $t \to +0$.

To prove Theorem 3 we need the following two lemmas. Since the proof of Lemma 3.1 is quite technical, we should give a complete proof for readers' convenience.

Lemma 3.1. *If $r, s \in N$, $0 \leqslant \lambda \leqslant 1$ and $s < \alpha < s + 2r$, then for $n \geqslant 2r + s + 1$, we have*

$$\| \delta_n^{2r+(\alpha-s)(\lambda-1)} B_{n,s}^{(2r)}(f) \|_\infty \leqslant C n^r \| \delta_n^{(\alpha-s)(\lambda-1)} f \|_\infty, \quad f \in C[0,1], \tag{27}$$

and

$$\begin{aligned} &\| \delta_n^{2r+(\alpha-s)(\lambda-1)} B_{n,s}^{(2r)}(f) \|_\infty \\ &\quad \leqslant C \| \delta_n^{2r+(\alpha-s)(\lambda-1)} f^{(2r)} \|_\infty, \quad f^{(2r-1)} \in A.C._{\text{loc}}. \end{aligned} \tag{28}$$

Proof. For $n \geqslant 2r + s + 1$ and $f \in C[0,1]$, let

$$F(x) = n^s \int_0^{1/n} \cdots \int_0^{1/n} f\left(x + \sum_{i=1}^{s} u_i\right) du_1 \cdots du_s, \quad x \in \left[0, 1 - \frac{s}{n}\right].$$

We notice

$$B_{n,s}(f, x) = \sum_{k=0}^{n-s} F\left(\frac{n-s}{n} \cdot \frac{k}{n-s}\right) p_{n-s,k}(x) = B_{n-s}\left(F\left(\frac{n-s}{n} t\right), x\right).$$

Hence, following [7, pp. 125–128] we have

$$B_{n,s}^{(2r)}(f, x) = \frac{(n-s)!}{(n-2r-s)!} \sum_{k=0}^{n-2r-s} \vec{\Delta}_{1/(n-s)}^{2r} F\left(\frac{n-s}{n} \cdot \frac{k}{n-s}\right) p_{n-2r-s,k}(x), \tag{29}$$

where

$$\begin{aligned} \vec{\Delta}_{1/(n-s)}^{2r} F\left(\frac{n-s}{n} \cdot \frac{k}{n-s}\right) &= \sum_{j=0}^{2r} (-1)^j \binom{2r}{j} F\left(\frac{n-s}{n} \cdot \frac{k+2r-j}{n-s}\right) \\ &= \sum_{j=0}^{2r} (-1)^j \binom{2r}{j} F\left(\frac{k+2r-j}{n}\right) \end{aligned}$$

and

$$B_{n,s}^{(2r)}(f,x) = \varphi^{-4r}(x) \sum_{i=0}^{2r} Q_i(x, n-s)(n-s)^i \times \sum_{k=0}^{n-s} F\left(\frac{n-s}{n} \cdot \frac{k}{n-s}\right)\left(\frac{k}{n-s} - x\right)^i p_{n-s,k}(x), \tag{30}$$

where $Q_i(x, n-s)$ is a polynomial in $(n-s)x(1-x)$ of degree $[(2r-i)/2]$ with nonconstant bounded coefficients. Therefore,

$$| \varphi^{-4r}(x) Q_i(x, n-s)(n-s)^i | \leqslant C\left(n\varphi^{-2}(x)\right)^{r+(i/2)} \quad \text{for } x \in E_{n-s} = \left[\frac{1}{n-s}, 1 - \frac{1}{n-s}\right]. \tag{31}$$

For the sake of convenience, denote $v = (\alpha - s)(1 - \lambda)$. To prove (27), let $f \in C[0, 1]$. For $x \in E_{n-s}^c = \left[0, \frac{1}{n-s}\right) \cup \left(1 - \frac{1}{n-s}, 1\right]$, we use (29) and Hölder inequality to have

$$\begin{aligned} | B_{n,s}^{(2r)}(f,x) | &\leqslant Cn^{2r} \| \delta_n^{-v} f \| \sum_{k=0}^{n-2r-s} \sum_{j=0}^{2r} \binom{2r}{j} n^s \int_0^{1/n} \cdots \int_0^{1/n} \delta_n^v \\ &\quad \times \left(\frac{k+2r-j}{n} + \sum_{i=1}^{s} u_i\right) du_1 \cdots du_s\, p_{n-2r-s,k}(x) \\ &\leqslant Cn^{2r} \| \delta_n^{-v} f \| \left(\sum_{k=0}^{n-2r-s} \sum_{j=0}^{2r} n^s \int_0^{1/n} \cdots \int_0^{1/n} \delta_n^{2r} \right. \\ &\quad \left. \times \left(\frac{k+2r-j}{n} + \sum_{i=1}^{s} u_i\right) du_1 \cdots du_s\, p_{n-2r-s,k}(x)\right)^{v/2r}. \end{aligned}$$

Clearly

$$\varphi^2\left(\frac{k+2r+s-j}{n}\right) \leqslant \varphi^2\left(\frac{k}{n-2r-s}\right) + \frac{C}{n}$$

and

$$\varphi^2\left(\frac{k}{n}\right) = \varphi^2(x) + (1-2x)\left(\frac{k}{n} - x\right) - \left(\frac{k}{n} - x\right)^2. \tag{32}$$

Using the above two inequalities, we conclude by the formula of polynomial, Cauchy–Schwartz inequality, (9.5.10) in [7] and the fact $\varphi^2(x) < \frac{1+s}{n}$ that

$$\begin{aligned} &\sum_{k=0}^{n-2r-s} \sum_{j=0}^{2r} n^s \int_0^{1/n} \cdots \int_0^{1/n} \varphi^{2r}\left(\frac{k+2r-j}{n} + \sum_{i=1}^{s} u_i\right) du_1 \cdots du_s\, p_{n-2r-s,k}(x) \\ &\leqslant \sum_{k=0}^{n-2r-s} \sum_{j=0}^{2r} \left(\varphi^2\left(\frac{k+2r+s-j}{n}\right) + \frac{s}{n}\right)^r p_{n-2r-s,k}(x) \end{aligned}$$

$$
\begin{aligned}
&\leqslant C \sum_{k=0}^{n-2r-s} \left(\varphi^{2r} \left(\frac{k}{n-2r-s} \right) + \frac{1}{n^r} \right) p_{n-2r-s,k}(x) \\
&\leqslant C \sum_{k=0}^{n-2r-s} \left(\left(\varphi^2(x) + (1-2x) \left(\frac{k}{n-2r-s} - x \right) \right.\right. \\
&\quad \left.\left. - \left(\frac{k}{n-2r-s} - x \right)^2 \right)^r + \frac{1}{n^r} \right) p_{n-2r-s,k}(x) \leqslant C n^{-r}.
\end{aligned}
$$

Thus, by the inequality $\delta_n^{2r}(x) \leqslant 2^{2r} \max\left\{\varphi^{2r}(x), n^{-r}\right\}$, we get for $x \in E_{n-s}^c$

$$
\begin{aligned}
\delta_n^{2r-v}(x) \mid B_{n,s}^{(2r)}(f,x) \mid \leqslant & \; C n^{-r+v/2} n^{2r} \parallel \delta_n^{-v} f \parallel \\
& \times \left(\sum_{k=0}^{n-2r-s} \sum_{j=0}^{2r} n^s \int_0^{1/n} \cdots \int_0^{1/n} 2^{2r} \right. \\
& \times \max \left\{ \varphi^{2r} \left(\frac{k+2r-j}{n} + \sum_{i=1}^{s} u_i \right), n^{-r} \right\} \\
& \left. \times \, du_1 \cdots du_s \, p_{n-2r-s,k}(x) \right)^{v/2r} \leqslant C n^r \parallel \delta_n^{-v} f \parallel .
\end{aligned}
$$

For $x \in E_{n-s}$, we apply (30), (31), Hölder inequality as well as Cauchy–Schwartz inequality to obtain

$$
\begin{aligned}
\mid B_{n,s}^{(2r)}(f,x) \leqslant & \; C \parallel \delta_n^{-v} f \parallel \sum_{i=0}^{2r} \left(\frac{n}{\varphi^2(x)} \right)^{r+(i/2)} \sum_{k=0}^{n-s} \left| \frac{k}{n-s} - x \right|^i n^s \\
& \times \int_0^{1/n} \cdots \int_0^{1/n} \delta_n^v \left(\frac{k}{n} + \sum_{i=1}^{s} u_i \right) du_1 \cdots du_s \, p_{n-s,k}(x) \\
\leqslant & \; C \parallel \delta_n^{-v} f \parallel \sum_{i=0}^{2r} \left(\frac{n}{\varphi^2(x)} \right)^{r+(i/2)} \\
& \times \left(\sum_{k=0}^{n-s} \left| \frac{k}{n-s} - x \right|^i p_{n-s,k}(x) \right)^{1-v/2r} \\
& \times \left(\sum_{k=0}^{n-s} \left| \frac{k}{n-s} - x \right|^i n^s \int_0^{1/n} \cdots \int_0^{1/n} \delta_n^{2r} \left(\frac{k}{n} + \sum_{i=1}^{s} u_i \right) \right. \\
& \left. \times \, du_1 \cdots du_s \, p_{n-s,k}(x) \right)^{v/2r}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\|\delta_n^{-v} f\| \sum_{i=0}^{2r}\left(\frac{n}{\varphi^2(x)}\right)^{r+(i/2)}\left(\sum_{k=0}^{n-s}\left(\frac{k}{n-s}-x\right)^{2i} p_{n-s,k}(x)\right)^{1/2}\\
&\times\left(\sum_{k=0}^{n-s}\left(n^s\int_0^{1/n}\cdots\int_0^{1/n}\delta_n^{2r}\right.\right.\\
&\left.\left.\times\left(\frac{k}{n}+\sum_{i=1}^{s}u_i\right)du_1\cdots du_s\right)^2 p_{n-s,k}(x)\right)^{v/4r}.
\end{aligned}
$$

Noticing $\varphi^2\left(\frac{k+s}{n}\right)\leqslant\varphi^2\left(\frac{k}{n-s}\right)+\frac{s(1+s)}{n}$, we get by (32), (9.5.10) in [7], and $\frac{1}{n}<2\varphi^2(x)$ that

$$
\begin{aligned}
&\sum_{k=0}^{n-s}\left(n^s\int_0^{1/n}\cdots\int_0^{1/n}\delta_n^{2r}\left(\frac{k}{n}+\sum_{i=1}^{s}u_i\right)du_1\cdots du_s\right)^2 p_{n-s,k}(x)\\
&\quad\leqslant C\sum_{k=0}^{n-s}\left(\varphi^2\left(\frac{k+s}{n}\right)+\frac{s}{n}\right)^{2r} p_{n-s,k}(x)\\
&\quad\leqslant C\sum_{k=0}^{n-s}\left(\varphi^2\left(\frac{k}{n-s}\right)+\frac{s(1+s)}{n}+\frac{s}{n}\right)^{2r} p_{n-s,k}(x)\\
&\quad\leqslant C\sum_{k=0}^{n-s}\left(\varphi^2(x)+\mid 1-2x\mid\left|\frac{k}{n-s}-x\right|\right.\\
&\qquad\left.+\left(\frac{k}{n-s}-x\right)^2+\frac{s(2+s)}{n}\right)^{2r} p_{n-s,k}(x)\\
&\quad\leqslant C\sum_{k=0}^{n-s}\left(\varphi^{4r}(x)+\left(\frac{k}{n-s}-x\right)^{2r}+\left(\frac{k}{n-s}-x\right)^{4r}+\frac{1}{n^{2r}}\right)p_{n-s,k}(x)\\
&\quad\leqslant C\varphi^{4r}(x).
\end{aligned}
$$

Therefore, using (9.5.10) in [7] again, we get for $x\in E_{n-s}$

$$
\begin{aligned}
&\delta_n^{2r-v}(x)\mid B_{n,s}^{(2r)}(f,x)\mid\\
&\quad\leqslant C\varphi^{2r-v}(x)\parallel\delta_n^{-v}f\parallel\sum_{i=0}^{2r}\left(\frac{n}{\varphi^2(x)}\right)^{r+(i/2)}\left(\frac{\varphi^2(x)}{n}\right)^{i/2}\varphi^v(x)\\
&\quad\leqslant Cn^r\parallel\delta_n^{-v}f\parallel,
\end{aligned}
$$

which proves (27).

Next we prove (28). Let $f^{(2r-1)} \in A.C_{\rm loc.}$ It is known that there is a function $G \in C[0,1]$ such that $G^{(s)} = f$. Following (9.4.3) in [7] we have

$$B_{n,s}^{(2r)}(f,x) = \frac{n^s(n-s)!}{n!} B_n^{(2r+s)}(G,x)$$
$$= \frac{n^s(n-s)!}{(n-2r-s)!} \sum_{k=0}^{n-2r-s} \vec{\Delta}_{1/n}^{2r+s} G\left(\frac{k}{n}\right) p_{n-2r-s,k}(x), \tag{33}$$

where

$$\vec{\Delta}_{1/n}^{2r+s} G\left(\frac{k}{n}\right) = \sum_{j=0}^{2r+s} (-1)^j \binom{2r+s}{j} G\left(\frac{k+2r+s-j}{n}\right).$$

For $1 \leqslant k \leqslant n-2r-s-1$, we have

$$n^{2r+s}\left|\vec{\Delta}_{1/n}^{2r+s} G\left(\frac{k}{n}\right)\right| \leqslant \sup_{\frac{k}{n} \leqslant \xi \leqslant \frac{k+2r+s}{n}} |\, G^{(2r+s)}(\xi)\, |$$
$$\leqslant \frac{\| \varphi^{2r-v} f^{(2r)} \|_\infty}{(k/n)^{r-v/2}\,((n-2r-s-k)/n)^{r-v/2}}. \tag{34}$$

Using the approach of Ditzian [2, pp. 281–282], one can easily see that for $k = 0$ or $n-2r-s$

$$n^{2r+s} \mid \vec{\Delta}_{1/n}^{2r+s} G(0) \mid \leqslant C n^{r-v/2} \| \varphi^{2r-v} f^{(2r)} \|_\infty \tag{35}$$

and

$$n^{2r+s} \mid \vec{\Delta}_{1/n}^{2r+s} G\left(\frac{n-2r-s}{n}\right) \mid \leqslant C n^{r-v/2} \| \varphi^{2r-v} f^{(2r)} \|_\infty . \tag{36}$$

Combining (33)–(36), and using again Hölder inequality and Lemma 3.2 in [2], we have for $x \in E_n$

$$\delta_n^{2r-v}(x) \mid B_{n,s}^{(2r)}(f,x) \mid$$
$$\leqslant C\varphi^{2r-v}(x) \| \varphi^{2r-v} f^{(2r)} \|_\infty$$
$$\times \sum_{k=0}^{n-2r-s} \frac{p_{n-2r-s,k}(x)}{((k+1)/n)^{r-v/2}\,((n-2r-s-k+1)/n)^{r-v/2}}$$
$$\leqslant C\varphi^{2r-v}(x) \| \varphi^{2r-v} f^{(2r)} \|_\infty$$
$$\times \left(\sum_{k=0}^{n-2r-s} \frac{p_{n-2r-s,k}(x)}{((k+1)/n)^{r}\,((n-2r-s-k+1)/n)^{r}} \right)^{1-v/2r}$$
$$\leqslant C\varphi^{2r-v}(x) \| \varphi^{2r-v} f^{(2r)} \|_\infty$$
$$\times \left(\sum_{k=0}^{n-2r-s} \left(\frac{p_{n-2r-s,k}(x)}{((k+1)/n)^{r}} + \frac{p_{n-2r-s,k}(x)}{((n-2r-s-k+1)/n)^{r}} \right) \right)^{1-v/2r}$$
$$\leqslant C \| \varphi^{2r-v} f^{(2r)} \|_\infty \leqslant C \| \delta_n^{2r-v} f^{(2r)} \|_\infty .$$

By (33), the first inequality in (34), and $G^{(2r+s)} = f^{(2r)}$, we get for $x \in E_n^c$

$$\delta_n^{2r-v}(x) \mid B_{n,s}^{(2r)}(f, x) \mid \leqslant C n^{-r+v/2} \parallel f^{(2r)} \parallel_\infty \leqslant C \parallel \delta_n^{2r-v} f^{(2r)} \parallel_\infty$$

which implies the assertion of Lemma 3.1. □

Lemma 3.2. *If $r \in N$, $0 < \beta < 2r$, $0 \leqslant t \leqslant \frac{1}{8r}$ and $rt < x < 1 - rt$, then*

$$\int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \delta_n^{-\beta} \left(x + \sum_{i=1}^{2r} u_i \right) du_1 \cdots du_{2r} \leqslant C t^{2r} \delta_n^{-\beta}(x). \tag{37}$$

Proof. Following [12, pp. 310–311] and using Hölder inequality we can deduce (37) easily. □

Proof of Theorem 3. Following (27), (28) and (37), and using the same method in [9, pp. 164–166] or [10, pp. 380–381], we can prove Theorem 3. We should omit the details. □

4. Equivalent result

In this final section, we discuss briefly the equivalent problem on pointwise approximation by these combinations of Bernstein operators with respect to $\omega_\varphi^{2r}(f, t)$ when $r \geqslant 2$ and $0 \leqslant \lambda < 1 - 1/r$, and obtain the following direct and equivalent results.

Theorem 4. *Let $r \in N$, $r \geqslant 2$, $0 \leqslant \lambda < 1 - 1/r$ and $J = \max\{j : r\lambda - 2r + j \leqslant 0,\ j \leqslant 2r - 1\}$. Then for all $f \in C[0, 1]$ and $n \in N$ with $n \geqslant G$, we have*

$$\mid B_n(f, r, x) - f(x) \mid \leqslant C \left(\sum_{i=r+1}^{J} \omega^i \left(f, \left(n^{-r} \varphi^{2(i-r)}(x) \right)^{1/i} \right) \right.$$
$$\left. + \, \omega_{\varphi^\lambda}^{2r} \left(f, n^{-1/2} \varphi^{1-\lambda}(x) \right) + n^{-r} \varphi^{2r(1-\lambda)}(x) \parallel f \parallel \right).$$

where G is a positive constant.

Theorem 5. *Let $r \in N$, $r \geqslant 2$, $0 \leqslant \lambda < 1 - 1/r$ and $0 < \alpha < \frac{2(r+1)}{2-\lambda}$. Then for all $f \in C[0, 1]$*

$$B_n(f, r, x) - f(x) = O\left(\left(n^{-1/2} \delta_n^{1-\lambda}(x) \right)^\alpha \right) \text{ as } n \to \infty$$
$$\Leftrightarrow \omega_{\varphi^\lambda}^{2r}(f, t) = O\left(t^\alpha \right) \text{ as } t \to +0.$$

The proof of Theorem 4 is similar to that of Theorem 2. And the proof of Theorem 5 is similar to that of Theorem 1.

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Appendix A. The detailed proof of Eq. (17)

For $n \geqslant s$ and $r \leqslant i \leqslant J$, we define

$$T_{n,i}(f,x) = \begin{cases} -\frac{1}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right) \vec{\Delta}^{i}_{|R_{n,i}(x)|^{1/i}} f(x) & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{(-1)^{i+1}}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right) \vec{\Delta}^{i}_{-|R_{n,i}(x)|^{1/i}} f(x) & \text{for } \frac{1}{2} < x \leqslant 1, \end{cases}$$

where $R_{n,i}(x) = B_{n,s}\left((t-x)^i, r, x\right)$, and $\vec{\Delta}^{i}_{t} f(x) = \sum_{k=0}^{i} (-1)^k \binom{i}{k} f(x+(i-k)t)$ for $0 \leqslant x \leqslant 1 - it$ and $\vec{\Delta}^{i}_{t} f(x) = 0$ otherwise.

For $n \geqslant \left(2JM_2^{1/r}\right)^{J/r}$ and $\frac{1}{n} \leqslant x \leqslant \frac{1}{2}$, by (10) and (11), $x + i \mid R_{n,i}(x) \mid^{1/i} \leqslant x + JM_2^{1/i}\left(\frac{\varphi(x)^{2i-2r}}{n^r}\right)^{1/i} \leqslant x + JM_2^{1/r}\frac{1}{n^{r/J}} \leqslant \frac{1}{2} + \frac{1}{2} = 1$. Following the definition above, we have

$$\begin{aligned} & T_{n,i}\left((t-x)^j, x\right) \\ & = -\frac{1}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right) \sum_{k=0}^{i} (-1)^k \binom{i}{k} \left(x + (i-k) \mid R_{n,i}(x) \mid^{1/i} - x\right)^j \\ & = -\frac{1}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right) \sum_{k=0}^{i} (-1)^k \binom{i}{k} (i-k)^j \mid R_{n,i}(x) \mid^{j/i} \\ & = -\frac{1}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right) \mid R_{n,i}(x) \mid^{j/i} \sum_{k=0}^{i} (-1)^k \binom{i}{k} (i-k)^j \\ & = \begin{cases} 0, & j < i, \\ -R_{n,i}(x), & j = i, \\ C_{i,j}\left(\operatorname{sgn} R_{n,i}(x)\right) \mid R_{n,i}(x) \mid^{j/i}, & j > i, \end{cases} \end{aligned}$$

where we have used the fact

$$\sum_{k=0}^{i} (-1)^k \binom{i}{k} (i-k)^j = \begin{cases} 0, & j < i, \\ i!, & j = i, \end{cases}$$

and

$$C_{i,j} = -\frac{1}{i!} \sum_{k=0}^{i} (-1)^k \binom{i}{k} (i-k)^j \quad \text{for } j > i.$$

For $n \geqslant \left(2JM_2^{1/r}\right)^{J/r}$ and $\frac{1}{2} < x \leqslant 1 - \frac{1}{n}$, by (10) and (11), $x - i \mid R_{n,i}(x) \mid^{1/i} \geqslant x - JM_2^{1/i}\left(\frac{\varphi(x)^{2i-2r}}{n^r}\right)^{1/i} \geqslant x - JM_2^{1/r}\frac{1}{n^{r/J}} > \frac{1}{2} - \frac{1}{2} = 0$. Following the definition above, we have

$$
\begin{aligned}
&T_{n,i}\left((t-x)^j, x\right)\\
&= \frac{(-1)^{i+1}}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right)\sum_{k=0}^{i}(-1)^k\binom{i}{k}\left(x-(i-k)\mid R_{n,i}(x)\mid^{1/i} - x\right)^j\\
&= \frac{(-1)^{i+1}}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right)\sum_{k=0}^{i}(-1)^k\binom{i}{k}(-1)^j(i-k)^j \mid R_{n,i}(x)\mid^{j/i}\\
&= \frac{(-1)^{i+j+1}}{i!}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}\sum_{k=0}^{i}(-1)^k\binom{i}{k}(i-k)^j\\
&= \begin{cases} 0, & j < i,\\ -R_{n,i}(x), & j = i,\\ C_{i,j}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}, & j > i, \end{cases}
\end{aligned}
$$

where we have also used the fact above and $C_{i,j} = \frac{(-1)^{i+j+1}}{i!}\sum_{k=0}^{i}(-1)^k\binom{i}{k}(i-k)^j$ for $j > i$.

Appendix B. The detailed proof of $A_n\left((t-x)^j, x\right) = 0$ for $j = 1, \ldots, J$

For $1 \leqslant j \leqslant r-1$, using (9), and the first lines in (17) and (19), we have

$$
\begin{aligned}
A_n\left((t-x)^j, x\right) &= B_{n,s}\left((t-x)^j, r, x\right) + \sum_{i=r}^{J}\Bigg\{T_{n,i}\left((t-x)^j, x\right)\\
&\qquad + \sum_{r \leqslant i < j_1 < \cdots < j_k \leqslant J} T_{n,i,j_1,\ldots,j_k}\left((t-x)^j, x\right)\Bigg\}\\
&= 0 + \sum_{i=r}^{J}\Bigg\{0 + \sum_{r \leqslant i < j_1 < \cdots < j_k \leqslant J} 0\Bigg\} = 0.
\end{aligned}
$$

For $r \leqslant j \leqslant J$, using (17) and (19), we have (note that, when $J = r$ the second sum does not exist in the following Eqs. (38) and (39), and when $j = r$, by the first line in (19), the second sum does not exist in the following Eq. (40).)

$$
\begin{aligned}
A_n\left((t-x)^j, x\right) &\equiv B_{n,s}\left((t-x)^j, r, x\right) + \sum_{i=r}^{J}\left\{T_{n,i}\left((t-x)^j, x\right)\right. \\
&\quad \left. + \sum_{r \leqslant i < j_1 < \cdots < j_k \leqslant J} T_{n,i,j_1,\ldots,j_k}\left((t-x)^j, x\right)\right\} \qquad (38)\\
&\equiv R_{n,j}(x) + \sum_{i=r}^{J} T_{n,i}\left((t-x)^j, x\right) \\
&\quad + \sum_{k=1}^{J-r}\sum_{i=r}^{J-k}\sum_{j_1=i+1}^{J-k+1}\cdots\sum_{j_k=j_{k-1}+1}^{J} T_{n,i,j_1,\ldots,j_k}\left((t-x)^j, x\right) \qquad (39)\\
&= R_{n,j}(x) + \sum_{i=r}^{j} T_{n,i}\left((t-x)^j, x\right) \\
&\quad + \sum_{k=1}^{j-r}\sum_{i=r}^{j-k}\sum_{j_1=i+1}^{j-k+1}\cdots\sum_{j_k=j_{k-1}+1}^{j} T_{n,i,j_1,\ldots,j_k}\left((t-x)^j, x\right) \qquad (40)\\
&= R_{n,j}(x) + \sum_{i=r}^{j} T_{n,i}\left((t-x)^j, x\right) \\
&\quad + \sum_{i=r}^{j-1}\sum_{j_1=i+1}^{j} T_{n,i,j_1}\left((t-x)^j, x\right) \\
&\quad + \sum_{i=r}^{j-2}\sum_{j_1=i+1}^{j-1}\sum_{j_2=j_1+1}^{j} T_{n,i,j_1,j_2}\left((t-x)^j, x\right) \\
&\quad + \cdots + \sum_{i=r}^{r+1}\sum_{j_1=i+1}^{r+2}\sum_{j_2=j_1+1}^{r+3}\cdots\sum_{j_{j-r-2}=j_{j-r-3}+1}^{j-1}\ \sum_{j_{j-r-1}=j_{j-r-2}+1}^{j} \\
&\quad \times T_{n,i,j_1,j_2,\ldots,j_{j-r-2},j_{j-r-1}}\left((t-x)^j, x\right) \\
&\quad + T_{n,r,r+1,r+2,\ldots,j-2,j-1,j}\left((t-x)^j, x\right) \qquad (41)\\
&= R_{n,j}(x) + T_{n,j}\left((t-x)^j, x\right) \\
&\quad + \sum_{i=r}^{j-1} T_{n,i}\left((t-x)^j, x\right) \sum_{i=r}^{j-1} T_{n,i,j}\left((t-x)^j, x\right) \\
&\quad + \sum_{i=r}^{j-2}\sum_{j_1=i+1}^{j-1} T_{n,i,j_1}\left((t-x)^j, x\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=r}^{j-2}\sum_{j_1=i+1}^{j-1} T_{n,i,j_1,j}\left((t-x)^j,x\right)\\
&+\sum_{i=r}^{j-3}\sum_{j_1=i+1}^{j-2}\sum_{j_2=j_1+1}^{j-1} T_{n,i,j_1,j_2}\left((t-x)^j,x\right)\\
&+\cdots+\sum_{i=r}^{r+1}\sum_{j_1=i+1}^{r+2}\sum_{j_2=j_1+1}^{r+3}\cdots\\
&\times\sum_{j_{j-r-2}=j_{j-r-3}+1}^{j-1} T_{n,i,j_1,j_2,\ldots,j_{j-r-2},j}\left((t-x)^j,x\right)\\
&+T_{n,r,r+1,r+2,\ldots,j-2,j-1}\left((t-x)^j,x\right)\\
&+T_{n,r,r+1,r+2,\ldots,j-2,j-1,j}\left((t-x)^j,x\right) \qquad (42)\\
=\;& R_{n,j}(x)-R_{n,j}(x)+\sum_{i=r}^{j-1} C_{i,j}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}\\
&-\sum_{i=r}^{j-1} C_{i,j}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}\\
&+\sum_{i=r}^{j-2}\sum_{j_1=i+1}^{j-1} C_{i,j_1,j}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}\\
&-\sum_{i=r}^{j-2}\sum_{j_1=i+1}^{j-1} C_{i,j_1,j}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}\\
&+\sum_{i=r}^{j-3}\sum_{j_1=i+1}^{j-2}\sum_{j_2=j_1+1}^{j-1} C_{i,j_1,j_2,j}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}\\
&-\cdots-\sum_{i=r}^{r+1}\sum_{j_1=i+1}^{r+2}\sum_{j_2=j_1+1}^{r+3}\cdots\\
&\times\sum_{j_{j-r-2}=j_{j-r-3}+1}^{j-1} C_{i,j_1,j_2,\ldots,j_{j-r-2},j}\left(\operatorname{sgn} R_{n,i}(x)\right)|R_{n,i}(x)|^{j/i}\\
&+C_{r,r+1,r+2,\ldots,j-2,j-1,j}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}\\
&-C_{r,r+1,r+2,\ldots,j-2,j-1,j}\left(\operatorname{sgn} R_{n,i}(x)\right)\mid R_{n,i}(x)\mid^{j/i}\\
=\;&0. \qquad (43)
\end{aligned}
$$

In the equalities above, Eq. (38) follows from the definition of A_n, Eq. (39) follows from the definition of the second sum in Eq. (38), and Eq. (40) is obtained using the first line of

(17) and (19). Eq. (41) is the detailed representation of $\sum_{k=1}^{j-r}$ in Eq. (40), where we note that

$$\sum_{i=r}^{r}\sum_{j_1=i+1}^{r+1}\cdots\sum_{j_{j-r}=J_{j-r-1}+1}^{j}T_{n,i,j_1,\ldots,j_{j-r}}\left((t-x)^j,x\right)=T_{n,r,r+1,\ldots,j}\left((t-x)^j,x\right).$$

In Eq. (42), when $j_1 = j-1$ we only have $i \leqslant j-2$, when $j_2 = j-1$ we only have $j_1 \leqslant j-2$ and $i \leqslant j-3$, and we can deduced the rest by analogy. Eq. (43) is obtained using the second and third lines in (17) and (19).

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